Law of Total Probability and Bayes’ Theorem

Problems 1-4. (Law of Total Probability and Bayes’ Theorem) Let $B_1$ and $B_2$ partition the sample space $\Omega$.

Suppose we are given the following information:

\[ P(B_1) = \frac{2}{3}, \quad P(B_2) = \frac{1}{3}, \quad P(A \mid B_1) = \frac{2}{3}, \quad \text{and} \quad P(A \mid B_2) = \frac{1}{2}. \]

1. Find $P(A)$.

2. Find $P(B_1 \mid A)$.

Suppose we are given the following information:

\[ P(B_1) = \frac{1}{5}, \quad P(B_2) = \frac{4}{5}, \quad P(A \mid B_1) = \frac{1}{3}, \quad \text{and} \quad P(A \mid B_2) = \frac{1}{2}. \]

3. Find $P(A)$.

4. Find $P(B_1 \mid A)$.

5-6. Suppose that we have two identical boxes: box 1 and box 2. Box 1 contains 5 red balls and 3 blue balls. Box 2 contains 2 red balls and 4 blue balls. A box is selected at random and exactly one ball is drawn from the box.
5. (Law of Total Probability) What is the probability that the ball is blue?

6. (Bayes’ Theorem) Given that the selected ball is blue, what’s the probability that it came from box 2?

Intermediate-Level Problems

7-9. You are selling a product in an area where 30% of the people live in the city; the rest live in
the suburbs. 20% of the city dwellers (urbanites) use your product; and 10% of the suburbanites use your product.

7. (Law of Total Probability) What fraction of the people in the area use your product?

8. (Law of Total Probability) You must choose between one of two sales strategies. The first will increase the fraction of city dwellers using your product from 20% to 25%; the second will increase the fraction of suburbanites using your product from 10% to 15%. Which strategy
9. (Bayes’ Theorem) What percentage of the people currently using your product are city dwellers?
10. (Law of Total Probability) A high school conducts random drug tests on its students. Of the student body, it is known that 8% use marijuana regularly; 17% use it occasionally; and 75% never use it. The testing regime is not perfect: regular marijuana users falsely test negative 5% of the time; occasional users falsely test negative 13% of the time; and non-users falsely test
positive 11% of the time. What percentage of the student body will test positive for marijuana use?

11-12. (Bayes’ Theorem) An inexpensive blood test can be used to test whether or not a person has a certain type of cancer. The test is not perfect: there is a 12% chance that a person who has the cancer will falsely test negative, and a 15% chance that a person who does not have the cancer will falsely test positive. More accurate (and more expensive) testing has shown that the
cancer is present in 8% of the tested population.

11. What is the probability that a person who tests negative has this type of cancer?

12. What is the probability that a person who tests positive has this type of cancer?
13. (Bayes’ Theorem) A box contains 3 coins. One coin has 2 heads and the other two are fair. A coin is chosen at random from the box and flipped. If the coin turns up heads, what is the probability that it is the two-headed coin? Is the answer 1/3?
14. (Bayes’ Theorem) An insurance company divides its policy holders into three categories: low risk, moderate risk, and high risk. The low-risk policy holders account for 60% of the total number of people insured by the company. The moderate-risk policy holders account for 30%, and the high-risk policy holders account for 10%. The probabilities that a low-risk, moderate-
risk, and high-risk policy holder will file a claim within a given year are respectively .01, .10, and .50. Given that a policy holder files a claim this year, what is the probability that the person is a high-risk policy holder?

15. (Law of Total Probability) A card is drawn from a standard deck of 52 cards and discarded (i.e. not replaced). A second card is drawn from the remaining deck of 51 cards. What is the probability that the second card is a spade?
16. (Bayes’ Theorem) A card is drawn from a standard deck of 52 cards and discarded (i.e. not replaced). A second card is drawn from the remaining deck of 51 cards. Given that the second card was a spade, what is the probability that the first card was also a spade?